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Rational Approximation with Real Zeros and Poles

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If we attempted to approximate the *real* continuous functions by polynomials with only *real* zeros, we would meet with dismal failure. A function, like $x^2 + 1$, with a positive minimum, cannot be approximated well at all, since (nonconstant) polynomials with real zeros have all their maxima nonnegative and all their minima nonpositive In fact the approximable functions are known to form a very small subclass of the continuous ones. (They are analytically continuable to be of finite order with real zeros.)

This situation changes dramatically, however, if we look to rational function, rather than polynomial, approximation. There have been many situations uncovered recently where rational functions perform much better than polynomials and this is one more such instance. We will see, namely, that rational functions with real zeros and poles approximate arbitrary real continuous functions arbitrarily well! Indeed much more is true and one does not need anywhere near this many zeros and poles.

Let S be any set of real numbers and denote by R_S the set of all rational functions having all zeros and poles in S. Also let $C[a, b]$ denote the space of *real* continuous functions under the usual sup-norm. We have then

THEOREM. *If S is dense in $[a, b]$ and, for some $\epsilon > 0$, $S - [a - \epsilon, b + \epsilon]$ is infinite, then R_S is dense in $C[a, b]$.*

Remarks. We note that these conditions are almost best possible, it being clearly necessary that S be dense in $[a, b]$ and also that $S - [a, b]$ be infinite. What is amusing is the counterexample which shows that something more is required. Let $[a, b] = [0, 1]$ and choose S as $[0, 1]$ together with the sequence $2^n/(2^n - 1)$; $n = 1, 2, 3, \ldots$ Since the linear function $l(x) = x - (2^n/(2^n - 1))$ satisfies $l(0)/l(1) = 2^n$, it follows that all the $r(x) \in R_S$, which do not vanish on $[0, 1]$ must satisfy $r(0)/r(1) = 2^k$, k an integer, and therefore the only nonvanishing functions which can be approximated must also satisfy this condition!

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Proof of the Theorem. We first reduce the problem to that of approximating positive functions in $C[a, b]$. For let $f(x)$ be any function of $C[a, b]$ and pick a nearby polynomial. We may then slightly perturb those zeros of this polynomial from $[a, b]$ to all lie in S. This new polynomial can be written as $p(x)\,q(x)$ where $q(x)$ has all its zeros $\in S$ (so that $q(x) \in R_S$) and where $p(x) > 0$ on $[a, b]$. The promised reduction is thereby achieved since we need now only approximate $p(x)$.

Taking logarithms, then, allows us to restate our result as the fact that the linear combinations $C + \sum_{k=1}^{n} V_k \log|x - x_k|$, V_k integers, C real, $x_k \in S - [a, b]$, are dense in $C[a, b]$. We prove in fact that this is the case with the added restriction $\sum_{k=1}^{n} V_k = 0$. So denote by L_S the collection of these expressions under this additional restriction.

Now let ξ be a limit point of S lying outside of $[a, b]$. We wish to reduce to the case of $\xi = \infty$, so suppose $\xi \neq \infty$ and map the reals by $y = 1/(x - \xi)$. This takes $[a, b]$ onto either $[1/(b - \xi), 1/(a - \xi)]$ or $[1/(a - \xi), 1/(b - \xi)]$. It also sends S onto a set T which *does have* ∞ as a limit point. Furthermore,

$$C + \sum_{k=1}^{n} V_k \log|x - x_k|$$

$$= \left(C - \sum_{k=1}^{n} V_k \log|y_k|\right) + \sum_{k=1}^{n} V_k \log|y - y_k|$$

so that L_S maps over to L_T and the reduction is complete.

We assume then that $\xi = \infty$ and proceed by induction on N to show, for any real C, that $Cx^N \in \bar{L}_S$. The case of $N = 0$ has been hypothesized and so we proceed to general N and we can assume that every polynomial of degree $< N$ already lies in $\bar{L}_S$. Choose x_1, x_2 large elements of S and note that $-\log|1 - (x/x_1)| + \log|1 - (x/x_2)| \in L_S$. Hence letting $x_2 \to \infty$ gives $-\log|1 - (x/x_1)| \in \bar{L}_S$ and substracting off the partial sum, this tells us that $\sum_{j \geqslant N} (1/j)(x/x_1)^j \in \bar{L}_S$. Now choose an integer V within 1 of CNx_1^N and with $|V| \leqslant |CNx_1^N|$, and conclude that $V \cdot \sum_{j \geqslant N} (1/j)((x/x_1))^j \in \bar{L}_S$. To check that this is close to Cx^N we have, writing $A = \text{Max}(|a|, |b|)$,

$$\left| V \cdot \sum_{j \geqslant N} \frac{1}{j}\left(\frac{x}{x_1}\right)^j - Cx^N \right| \leqslant 1 \cdot \frac{1}{N}\left(\frac{|x|}{|x_1|}\right)^N + \sum_{j > N} \frac{|V|}{j}\left(\frac{|x|}{|x_1|}\right)^j$$

$$\leqslant \left(\frac{A}{|x_1|}\right)^N + |C|\, A^N \sum_{j > N} \left(\frac{A}{|x_1|}\right)^{j-N}$$

$$= \left(\frac{A}{|x_1|}\right)^N + \frac{|C|\, A^{N+1}}{|x_1| - A}$$

and this can be made arbitrarily small by taking x_1 large enough. The induction is complete.

Having shown, then, that $\bar{L}_S$ contains all (real) polynomials we simply invoke Weierstrass' theorem to complete our proof.